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The basic problem addressed under this research is the development of a scheme for "deconvolving" a set of correlated signals in a multichannel scenario so as to assure signal-to noise enhancement. The author addressed this problem during the previous work period via the development of the Karhunen-Loeve transformation. The original purpose of this phase was to continue this linear of approach and to some extent this was achieved. However it became necessary to alter the course temporally for reasons to be discussed below. We really cannot continue in any serious way until we can evaluate numerically the highly oscillatory integrals that are an integral part of the analysis. The reason is that asymptotic evaluations of these integrals is simply not powerful enough to yield accurate numerical results; for the same reason expansions in special functions are also not very effective. Thus we are forced to consider sophisticated numerical techniques to evaluate the integrals; this is really a step forward because computers are not so fast that one can almost gain the speed of an FFT (which is known to be a reasonably inaccurate way to evaluate oscillatory integrals). To this end I have decided to concentrate upon the development of accurate numerical evaluation of zero and first order Hankel transforms as the majority of the integrals need to secure understanding of the Karhunen-Loeve approach to the averaging process require such integrals. In addition, I have worked out a numerical scheme for evaluation of finite range Fourier integrals as such integrals appear in my approach to the laser doppler in the new FM analysis which supersedes the old George-Lumley approach. Thus there are two sections entitled: a) Filon trapezoidal schemes for Hankel transforms of orders zero and one b) Numerical evaluation of Fourier integrals: Filon quadrature versus the FFT. I should like to point out that Barbara Sandler has assisted me in the programming of item a. In addition I have written a paper under the aegis of the contract; it is the first item. Now that I am in possession of the necessary numerical techniques I intend to employ them on the original problem during the next phase.

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Second-order intensity statistics of a coherent signal in the presence of a random background

R. BARAKAT†

The statistics of a harmonic signal (coherent component) mixed with a random background (incoherent component) of a specified spectra profile (power spectrum) is still a problem of interest. The purpose of the present paper is to study the second-order intensity statistics of such a signal/background situation using the generalized Karhunen-Loevé expansion developed for use in photon counting and laser speckle.

1. Introduction

In a classic paper written several years after the actual work (done during World War II), Kac and Siegert (1947) determined the exact first-order statistics (i.e. probability density and moments) of a square law detector for a gaussian random field, and for an harmonic signal buried in a gaussian random field. Their approach involves the construction of a homogeneous integral equation whose kernel is the covariance function of the random field using what is now termed the Karhunen-Loevé expansion (Selin 1965, Thomas 1964); although we can also term the construction the Kac-Siegert expansion in as much as they derived it independently. The eigenvalues determine the probability density function of the detected intensity (square of the field amplitude), while both eigenvalues and eigenfunctions are needed for the probability density of the intensity of the signal and field. Emerson (1953) also discussed the problem from an alternative viewpoint, using the method of cumulants to avoid solving the associated homogeneous integral of Kac and Siegert. For further work on the problem, see Slepian (1958). As Mayer and Middleton (1954) have pointed out, the original expansion method of Kac-Siegert is not general enough to handle higher-order statistics such as product moments. However, Kac-Siegert also presented another method of solution, the 'direct' method, which is capable of handling higher-order statistics. The direct method requires an appropriate transformation to express the output in terms of the input; the statistics of the output are then determined by suitable additional transformations with respect to the original input statistics. Mayer' and Middleton exploited this approach to determine the higher-order statistics of the output due to a square law detector for both narrow-band and broad-band inputs. Kac-Siegert only considered the broad-band situation.

The purpose of the present paper is to restudy the above problems for the second-order intensity statistics using a generalization of the *original* Kac-Siegert expansion approach. I employ two point detectors to interrogate the random input (with and without the harmonic signal present). These detectors operate for the same time interval but are delayed relative to each other by a variable time. The analysis is

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performed via an expansion of the random field over these two disjoint time intervals, using a generalization of the Karhunen-Loevé series developed for similar problems in photon counting (Jakeman 1970, Blake and Barakat 1973, Barakat and Blake 1980) and laser speckle (Barakat and Blake 1978). In this approach, a homogeneous integral equation is constructed over the two *disjoint* time intervals wherein the two detectors operate. The eigenvalues and eigenfunctions (which obey an unusual orthogonality condition) are evaluated and used to fabricate a double generating function; from this double generating function the various product moments of the integrated intensities can be obtained by differentiation. Analysis is confined to the narrow-band situation, although there is no difficulty in studying the broad-band situation. I feel that the approach via the generalized Karhunen-Loevé expansion offers a more satisfying physical picture than the direct method in as much as the detector time intervals and delays are an inherent part of the analysis via the associated disjoint integral equation.

2. Formal solution

The complex-value $U(t)$ of the total disturbance is the sum of a deterministic (or coherent) component $U_c(t)$ and a random background term $U_b(t)$

$$U(t) = U_c(t) + U_b(t) \tag{1}$$

The coherent component is given by

$$U_c(t) = \xi_c \exp(-i\omega_c t) \tag{2}$$

where ξ_c is a constant. The random background $U_b(t)$ is taken to be a zero-mean, complex-valued, spatially stationary gaussian random process, i.e.

$$U_b(t) = U_b^{(r)}(t) + i U_b^{(i)}(t) \tag{3}$$

where $U_b^{(i)}(t)$ is the stochastic Hilbert transform of $U_b^{(r)}(t)$. Both $U_b^{(r)}$ and $U_b^{(i)}$ are real-valued. Now $U_b^{(r)}(t)$ and $U_b^{(i)}(t)$ have the same gaussian probability density function (PDF); also

$$\langle U_b^{(r)}(t) \rangle = \langle U_b^{(i)}(t) \rangle = 0 \tag{4}$$

$$\langle U_b^{(r)}(t_1) U_b^{(r)}(t_2) \rangle = \langle U_b^{(i)}(t_1) U_b^{(i)}(t_2) \rangle = \frac{\sigma^2}{2} r_b(t_1 - t_2) \tag{5}$$

$$\langle U_b^{(r)}(t) U_b^{(i)}(t) \rangle \equiv 0 \tag{6}$$

where σ_b^2 is the variance of $U_b(t)$ and $r_b(t_1 - t_2)$ is the corresponding correlation function, $0 \le |r_b| \le 1$. Since $U_b(t)$ is a gaussian random process, it is completely characterized by its mean, variance, and correlation function.

The disturbance $U(t)$ is interrogated by two-point detectors, the first detector operating during the time interval $(-T/2, T/2)$ and the second during the time interval $(\tau - T/2, \tau + T/2)$ where τ is the variable time delay. The integrated intensities Ω_j are

$$\Omega_1 = \int_{-T/2}^{T/2} |U_c(t) + U_b(t)|^2 \, dt$$

$$\Omega_2 = \int_{\tau-T/2}^{\tau+T/2} |U_c(t) + U_b(t)|^2 \, dt \tag{7}$$

We can expand $U_b(t)$ in a generalized Karhunen–Loevé series (Jakeman 1970, Blake and Barakat 1973, Barakat and Blake 1978, 1980) over the disjoint intervals

$$A_1 \equiv (-T/2, T/2), \quad A_2 \equiv (\tau - T/2, \tau + T/2) \tag{8}$$

so that

$$U_b(t) = \begin{cases} \sum_{k=0}^{\infty} U_k \psi_k(t), & t \in A_1 \text{ and } A_2 \\ \\ 0, & t \notin A_1 \text{ and } A_2 \end{cases} \tag{9}$$

The following conditions are to be satisfied.

(1) The $\{U_k\}$ are random coefficients, independent of t; and the uncorrelated gaussian random variables (hence, independent random variables)

$$\langle U_k U_l^* \rangle = \sigma_k^2 \delta_{kl} \tag{10}$$

where $\{\sigma_k\}$ are, as yet, unknown, real non-negative constants. It is essential that $U_b(t)$ be gaussian, for if it is not then the expansion coefficients $\{U_k\}$ will not be statistically independent, although they will still be uncorrelated.

(2) The deterministic functions $\{\psi_k(t)\}$ are to form a complete orthonormal set over both A_1 and A_2.

The precise statement of the orthogonality condition is quite unusual. Consider the weighted sum of the integrated intensities: $\lambda_1 \Omega_1 + \lambda_2 \Omega_2$ where λ_1, λ_2 are arbitrary real, non-negative parameters appearing in the two-fold generating function of the integrated background intensities $\Omega_j^{(b)}$

$$Q_b(\lambda_1, \lambda_2) = \langle \exp(-\lambda_1 \Omega_2^{(b)}) \rangle \tag{11}$$

We have

$$\lambda_1 \Omega_1 + \lambda_2 \Omega_2 = \oint |U_b(t)|^2 \, dt + \oint |U_c(t)|^2 \, dt + \oint U_b(t) U_c^*(t) \, dt + \oint U_b^*(t) U_c(t) \, dt \tag{12}$$

where

$$\oint \equiv \lambda_1 \int_{-T/2}^{T/2} + \lambda_2 \int_{\tau-T/2}^{\tau+T/2} \tag{13}$$

We now require

$$\oint \psi_k(t) \psi_l^*(t) \, dt = \delta_{kl} \tag{14}$$

The first term on the right-hand side of (12) can be evaluated using (9) so that

$$\lambda_1 \Omega_1^{(b)} + \lambda_2 \Omega_2^{(b)} = \sum_{k=0}^{\infty} |U_k|^2 \tag{15}$$

The second term yields

$$\oint |U_c(t)|^2 \, dt = 2|\xi_c|^2 T(\lambda_1 + \lambda_2) \tag{16}$$

The interaction integrals (third and fourth terms) are

$$\oint U_b(t) U_c^*(t) \, dt = \xi_c^* \sum_{k=0}^{\infty} U_k \oint \phi_i(t) \exp(i\Delta t) \, dt$$

$$\oint U_b^*(t) U_c(t) \, dt = \xi_c \sum_{k=0}^{\infty} U_k^* \oint \phi_i^*(t) \exp(-i\Delta t) \, dt \tag{17}$$

Here

$$\Delta \equiv (\omega_c - \omega_0) \tag{18}$$

is the frequency offset describing the position of the signal with respect to the maximum of the background power spectrum. Consequently (12) reduces to

$$\lambda_1 \Omega_1 + \lambda_2 \Omega_2 = \sum_{k=0}^{\infty} |U_k|^2 + 2|\xi_c|^2 T(\lambda_1 + \lambda_2)$$

$$+ \xi_c \sum_{k=0}^{\infty} U_k^* G_k(\lambda_1, \lambda_2) + \xi_c^* \sum_{k=0}^{\infty} U_k G_k^*(\lambda_1, \lambda_2) \tag{19}$$

where

$$G_k(\lambda_1, \lambda_2) \equiv \oint \phi_k(t) \exp(i\Delta t) \, dt \tag{20}$$

To evaluate the unknown constants σ_k in (10), we must construct an integral equation whose kernel is the correlation function of $U_b(t)$. The integral equation is

$$\left(\lambda_1 \int_{-T/2}^{T/2} + \lambda_2 \int_{\tau-T/2}^{\tau+T/2} \right) \tau_b(t_1 - t_2) \psi_i(t_2) \, dt_2 = \left(\frac{\sigma_i}{\sigma} \right)^2 \psi_i(t_1) \tag{21}$$

The correlation function $r_b(t_1 - t_2)$ of the random background field is given by

$$r_b(t_1 - t_2) = \exp\{(i\omega_0(t_1 - t_2)\} g(t_1 - t_2) \tag{22}$$

where $g_b(t_1 - t_2)$ is the correlation function centred at zero frequency and ω_0 is the frequency at which the power spectrum (lineshape) of the background radiation is a maximum. If we set

$$\phi_i(t) = \psi_i(t) \exp(i\omega_0 t) \tag{23}$$

then (15) reads

$$\left(\lambda_1 \int_{-T/2}^{T/2} + \lambda_2 \int_{\tau-T/2}^{\tau+T/2} \right) g_b(t_1 - t_2) \phi_i(t_2) \, dt_2 = \left(\frac{\sigma_i}{\sigma} \right)^2 \phi_i(t_1) \tag{24}$$

independent of ω_0. This is the basic integral equation for determining $(\sigma_l/\sigma)^2$; it is not of the standard Fredholm type because of the presence of two disjoint domains of integration. A second difficulty is that the eigenvalues σ_l^2/σ are implicit functions of λ_1 and λ_2.

The double generating function of the random background, $Q_b(\lambda_1, \lambda_2)$, can be expressed as an infinite product. From (11) (with $U_c(t) \equiv 0$ for the moment) we have

$$Q_b(\lambda_1, \lambda_2) = \int_0^\infty \cdots \int_0^\infty f[\{U_k\}] \exp\left(-\lambda_1 \Omega_1^{(b)} - \lambda_2 \Omega_2^{(b)}\right) \prod_{k=0}^\infty \mathrm{d}^2 U_k \qquad (25)$$

where $\mathrm{d}^2 U_k = \mathrm{d} U_k^{(r)} \mathrm{d} U_k^{(i)}$, and $f[\{U_k\}]$ is the joint probability density function of the statistically independent gaussian random variables

$$f[\{U_k\}] = \prod_{k=0}^\infty \frac{1}{\pi \sigma_k^2} \exp\left\{-|U_k|^2/\sigma_k^2\right\} \qquad (26)$$

Upon substituting (15) into the integrand, we encounter a standard gaussian integral; the final result is

$$Q_b(\lambda_1, \lambda_2) = \prod_{k=0}^\infty [1 + \sigma_k^2(\lambda_1, \lambda_2)]^{-1} \qquad (27)$$

Now for the double generating function with the coherent signal included. Given (11) and (19), it follows that

$$Q(\lambda_1, \lambda_2) = \exp\left\{-|\xi_c \subseteq^2 T(\lambda_1 + \lambda_2)\right\} \prod_{k=0}^\infty Q_k(\lambda_1, \lambda_2) \qquad (28)$$

where

$$Q(\lambda_1, \lambda_2) \equiv \frac{1}{\pi \sigma_k^2} \iint_{-\infty}^\infty \exp\left\{-(1 + \sigma_k^{-2})|U_k|^2\right\} \exp\left\{-\xi_q^* G_k^* U_k - \xi_q G_k U_k^*\right\} \mathrm{d}^2 U_k \qquad (29)$$

To evaluate Q_k, we note that since $U_k = V_k + iW_k$, then

$$\xi_c^* G_k^* U_k + \xi_c G_k U_k^* = a_k V_k + ib_k W_k \qquad (30)$$

where

$$a_k = \xi_c^* G_k^* + \xi_c G_k$$
$$b_k = \xi_c^* G_k^* - \xi_c G_k \qquad (31)$$

Consequently

$$Q_k = \frac{1}{\pi \sigma_k^2} \int_{-\infty}^\infty \exp\left\{-(1 + \sigma_k^{-2})V_k^2 - a_k V_k\right\} \mathrm{d} V_k \int_{-\infty}^\infty \exp\left\{-(1 + \sigma_k^{-2})W_k^2\right\} \mathrm{d} W_k \qquad (32)$$

Both integrals are known

$$\int_{-\infty}^{\infty} \exp\left\{-(1+\sigma_k^{-2})V_k^2 - a_k V_k\right\} \mathrm{d}V_k = \frac{\sqrt{\pi}}{(1+\sigma_k^{-2})^{1/2}} \exp\left[\frac{a_k^2}{4(1+\sigma_k^{-2})}\right] \quad (33)$$

$$\int_{-\infty}^{\infty} \exp\left\{-(+\sigma_k^2)W_k^{-2}\right\} \cos b_k W_k \, \mathrm{d}W_k = \frac{\sqrt{\pi}}{(1+\sigma_k^2)^{1/2}} \exp\left(\frac{a_k^2 - b_k^2}{4(1+\sigma_k^{-2})}\right) \quad (34)$$

Thus, $Q_k(\lambda_1, \lambda_2)$ reduces to

$$Q_k(\lambda_1, \lambda_2) = \frac{1}{(1+\sigma_k^2)} \exp\left[\frac{|\xi_c G_k|^2 \sigma_k^2}{(1+\sigma_k^2)}\right] \quad (35)$$

Consequently, the double generating function for the background/signal is

$$Q(\lambda_1, \lambda_2) = \exp\left\{-|\xi_c|^2 T(\lambda_1 + \lambda_2)\right\} \prod_{k=0}^{\infty} (1+\sigma_k^2)^{-1} \prod_{k=0}^{\infty} \exp\left\{\frac{|\xi_c G_k|^2 \sigma_k^2}{(1+\sigma_k^2)}\right\}$$

$$\equiv Q_c(\lambda_1, \lambda_2) Q_b(\lambda_1, \lambda_2) Q_{bc}(\lambda_1, \lambda_2) \quad (36)$$

where Q_c, Q_b are the generating functions for the coherent, background components respectively and Q_{bc} is the generating function for the interaction between background and signal.

The product moments of the integrated intensities can be obtained by differentiation of $Q(\lambda_1, \lambda_2)$

$$\langle \Omega_1^p \Omega_2^q \rangle = (-1)^{p+q} \frac{\partial^{p+q}}{\partial \lambda_1^p \partial \lambda_2^q} Q(\lambda_1, \lambda_2)\big|_{\lambda_1 = \lambda_2 = 0} \quad (37)$$

This completes the general solution.

3. Product moments of Ω, small T regime

It is obvious that the larger is T, the greater is the smoothing of the field amplitudes. Consequently, we want to employ as small a T as possible. Assume that T is small compared with the distance over which the background field correlation function decays to its $\exp^{-1}$ value. Under this condition, only the $k=0$ terms in Q_b and Q_{bc} contribute; all $k>0$ terms are essentially negligible. It has been shown that (Jakeman 1970, Blake and Barakat 1971, Barakat and Blake 1978, 1980)

$$\sigma_0^2(\lambda_1, \lambda_2) = \sigma^2 T(\lambda_1 + \lambda_2) + \sigma^4 T^2(1 + |g(t_1 - t_2)|^2)\lambda_1 \lambda_2 \quad (38)$$

The G_0 function can be evaluated by the mean value theorem for integrals. The result is

$$G_0 = b(\lambda_1 T + \lambda_2 T) \exp(i|t_1 - t_2|) \quad (39)$$

where $b = \phi_0(o)$.

Upon carrying out the necessary manipulation

$$Q(\lambda_1, \lambda_2) = (1+\sigma_0^2)^{-1} \exp\left\{-|\xi_c|^2 T(\lambda_1 + \lambda_2)\right\} \exp\left\{\frac{|\xi_c G_0|^2 \sigma_0^2}{(1+\sigma_0^2)}\right\} \quad (40)$$

The product moments of the integrated intensities were obtained by differentiation of $Q(\lambda_1, \lambda_2)$. I employed the symbol manipulation program, SMP, using a VAX computer to effect the differentiations. The first few product moments are

$$\langle \Omega_1 \rangle = \langle \Omega_2 \rangle = \sigma^2 + |\xi_c|^2 \tag{41}$$

$$\langle \Omega_1 \Omega_2 \rangle = \sigma^4 [1 + |g(\tau)|^2] + 2\sigma^2 |\xi_c|^2 + |\xi_c|^4 \tag{42}$$

REFERENCES

BARAKAT, R., and BLAKE, J., 1978, Second order statistics of speckle patterns observed through finite-size scanning apertures. *Journal of Optical Society of America.* **68**, 1217-1224; 1980, Theory of photoelectron counting statistics, an essay. *Physics Report*, **60**, 225-340.

BLAKE, J., and BARAKAT, R., 1973, Two-fold photoelectron counting statistics: The clipped correlation function. *Journal of Physics*, **6A**, 1196-1210.

EMERSON, R., 1993, First probability densities for receivers with square law detectors. *Journal of Applied Physics*, **24**, 1168-1176.

JAKEMAN, E., 1970, Theory of optical spectroscopy by digital autocorrelation of photon counting fluctuations. *Journal of Physics*, **A3**, 201-215.

KAC, M., and SIEGERT, A., 1947, On the theory of noise in radio receivers with square law detectors. *Journal of Applied Physics*, **18**, 383-397.

MAYER, M., and MIDDLETON, D., 1954, On the distributions of signals and noise after rectification and filtering. *Journal of Applied Physics.* **25**, 1037-1052.

SELIN, I., 1965, *Detection Theory* (Princeton, NJ: Princeton University Press). Chap. 3.

SLEPIAN, D., 1958, Fluctuations of random noise powers. *Bell System Technical Journal.* **37**, 163.

THOMAS, J., 1969, *Statistical Communication Theory* (New York: Wiley). Chap 6.

5. PRODUCT MOMENTS: BACKGROUND/SIGNAL

The product moments of the integrated intensities were obtained by differentiation of $Q(\lambda_1, \lambda_2)$, Eq. (4.3), according to Eq. (3.16); I employed the symbol manipulation program, SMP, using a VAX computer to effect the differentiations. The first few product moments are:

$$\langle \Omega_1 \rangle = \langle \Omega_2 \rangle = \sigma^2 + |\xi_c|^2 \tag{5.1}$$

$$\langle \Omega_1 \Omega_2 \rangle = \sigma^4[1 + g^2(\tau)] + 2\sigma^2|\xi_c|^2 + |\xi_c|^4 \tag{5.2}$$

$$\langle \Omega_1^2 \Omega_2 \rangle = \langle \Omega_1 \Omega_2^2 \rangle = 2\sigma^6[1 + 2g^2(\tau)]$$
$$+ \sigma^4|\xi_c|^2[4 + 2g^2(\tau) + b + 2b\cos\Delta\tau] + 3\sigma^2|\xi_c|^4 + |\xi_c|^6 \tag{5.3}$$

$$\langle \Omega_1^2 \Omega_2^2 \rangle = 4\sigma^8[1 + 4g^2(\tau) + g^4(\tau)]$$
$$+ \sigma^6|\xi_c|^2\{4 + 8g^2(\tau)[1 + b\cos\Delta\tau] + 8b\}$$
$$+ \sigma^4|\xi_c|^4[2 + g^2(\tau) + 4b + 8b\cos\Delta\tau]$$
$$+ 4\sigma^2|\xi_c|^6 + |\xi_c|^8 . \tag{5.4}$$

Note that the frequency offset Δ only appears in $\langle \Omega_1^2 \Omega_2 \rangle$ and $\langle \Omega_1^2 \Omega_2^2 \rangle$ and not in $\langle \Omega_1 \Omega_2 \rangle$.

In the homodyne case where the frequency of the signal coincides with the maximum of the power spectrum of the background, then Δ vanishes and the product moments then depend upon $g(\tau)$ only.

FILON TRAPEZOIDAL SCHEMES FOR HANKEL TRANSFORMS

OF ORDERS ZERO AND ONE

Abstract

Algorithms for evaluating zero order and first order Hankel transforms using Filon quadrature philosophy are developed in the context of a trapezoidal approximation rather than of a Simpson's rule approximation previously discussed. Unlike the Filon/Simpson algorithm previously developed, the Filon/trapezoidal algorithm tends to saturate, in that increasing the number of quadrature points does not materially increase the accuracy. Numerical examples are given and discussed.

1. Introduction

Previous papers [1,2] were devoted to the numerical evaluation of Hankel transforms of orders zero and one.

$$H(r) = \int_a^b h(p) J_n(rp) p\, dp \tag{1.1}$$

using Filon quadrature philosophy. In [1], the Filon approach is outlined in some detail for Eq 1.1 with $n = 0$. There the slowly varying part of the integrand, $h(p)$, is approximated by a quadratic function over the basic quadrate panel. For $n = 1$, see [2]. It was necessary to consider $\bar{h}(p) \equiv p h(p)$ as the basic function to be expressed as a quadratic. As with Filon's original approach to Fourier integrals, the errors incurred in Eq. 1.1 are proportional to the derivatives of $h(p)$ and $\bar{h}(p)$ themselves rather than to the whole integrand, hence are relatively independent of r.

In many areas, we do not require great accuracy for the Hankel transforms, but need to maintain a given accuracy more or less uniformly, independent of the magnitude of r. The purpose of the present communication is to develop the trapezoidal version of the Filon-Hankel approach. In [1,2], the scheme is really a generalization of Simpson's rule, since we are employing a quadratic approximation of $h(p)$ and $\bar{h}(p)$ over the panels. The present approach is essentially a generalization of the trapezoidal quadrature scheme, since we are approximating $h(p)$ and $\bar{h}(p)$ by a linear function over the panels.

2. Trapezoidal Algorithm/Zero-Order Transform

Consider the integral

$$H_k(r) = \int_{p_k}^{p_{k+1}} h(p) J_0(rp) p \, dp \tag{2.1}$$

for two points p_{k+1} and p_k, where

$$p_{k+1} - p_k = \delta$$

The points p_k, where $k = 0, 1, \ldots, N$, are a subset of $[a, b]$. Approximate $h(p)$ by a straight line between p_k and p_{k+1}:

$$h(p) = A + Bp \tag{2.2}$$

It follows that

$$A = \frac{1}{\delta}(p_{k+1} h_k - p_k h_{k+1}) \tag{2.3}$$

$$B = \frac{1}{\delta}(h_{k+1} - h_k) \tag{2.4}$$

where $h(p_k) \equiv h_k$.

Upon setting $u = h(p)$ and $dv = J_0(rp) p \, dp$, we integrate $H_k(r)$ by parts; the end result is

$$H_k(r) = \frac{1}{r}\left[h_{k+1} J_1(rp_{k+1}) p_{k+1} - h_k J_1(rp_k) p_k\right] - \frac{1}{r^3}\left[h'_{k+1} \$_0(rp_{k+1}) - h'_k \$_0(rp_k)\right] \tag{2.5}$$

where we have used

$$\int^x y J_0(y) dy = x J_1(x) \tag{2.6}$$

The numerical evaluation of the function

$$\$_0(x) \equiv \int_0^x y J_1(y) dy \tag{2.7}$$

is discussed at some length in the Appendix of [1], to which we refer.

Since

$$h'_{k+1} = h'_k = \frac{1}{\delta}(h_{k+1} - h_k) \tag{2.8}$$

then Eq. 2.5 reduces to

$$H_k(r) = \frac{1}{r}\left[h_{k+1} J_1(rp_{k+1}) - h_k J_1(rp_k)\right] - \frac{1}{\delta r^3}(h_{k+1} - h_k)\left[\$_0(rp_{k+1}) - \$_0(rp_k)\right] \tag{2.9}$$

3

To evaluate the integral over $[a, b]$, i.e., Eq. 1.1, divide $[a, b]$ thusly

$$b = a + N\delta \tag{2.10}$$

Hence

$$H(r) = \sum_{k=0}^{N} H_k(r) \tag{2.11}$$

The final result is

$$H(r) = \frac{1}{r}\left[h_{N+1}J_1(rp_N) - h_0 J_1(rp_0)\right] - \frac{1}{\delta r^3}\sum_{k=0}^{N}(h_{k+1} - h_k)\left[\$_0(rp_{k+1}) - \$_0(rp_k)\right] \tag{2.12}$$

This is the basic formula for the Filon/trapezoidal scheme for zero-order Hankel transforms.

3. Numerical Example for the Zero-order Case

As with the Filon/Simpson algorithm for the zero-order Hankel transform, we consider as in [1]

$$h(p) = \frac{2}{\pi}\left[\text{arcos}(p) - p(1 - p^2)^{\frac{1}{2}}\right], \quad 0 \le p \le 1 \tag{3.1}$$

$$H(r) = \left[\frac{2J_1(r)}{r}\right]^2, \quad 0 \le r < \infty \tag{3.2}$$

as our test case.

Unlike the Filon/Simpson algorithm, the Filon/trapezoidal algorithm tends to saturate, in that increasing N does not materially increase the accuracy of the quadrature. Of course this is not surprising, because we are now approximating $h(p)$ by straight lines rather than by quadratic. In the context of our numerical example, perhaps the best way to see this is to fix r while varying N, examining the absolute error

$$|H(r)_{\text{exact}} - H(r)_{\text{computed}}| \tag{3.3}$$

Some typical values of the absolute error are displayed in Table 1. This table does not require any detailed comment.

In evaluating the various J-Bessel functions, we employed Mason's algorithm [3], which is probably the most accurate currently available.

4. Trapezoidal Algorithm/First-Order Transform

Analagous to Eq. 2.1, we write

$$H_k(r) = \int_{p_k}^{p_{k+1}} \bar{h}(p) J_1(rp) dp \qquad (4.1)$$

where

$$\bar{h}(p) \equiv h(p)p \qquad (4.2)$$

Now set $u = \bar{h}(p)$ and $dv = J_1(rp)dp$. Upon integrating by parts, we obtain

$$H_k(r) = -\frac{1}{r}\left[\bar{h}(p_{k+1})J_0(rp_{k+1}) - \bar{h}(p_k)J_0(rp_k)\right]$$
$$+ \frac{1}{r^2}\left[\bar{h}'(p_{k+1})R_0(rp_{k+1}) - \bar{h}'(p_k)R_0(rp_k)\right] \qquad (4.3)$$

where we have used the indefinite integral

$$\int^x J_1(y)dy = -J_0(x) \qquad (4.4)$$

and

$$R_0(x) \equiv \int_0^x J_0(y)dy \qquad (4.5)$$

See Appendix A of [2] for the numerical evaluation of this function.

As in Section 2, we can sum the various panels, leading to

$$H(r) = -\frac{1}{r}\left[\bar{h}_N J_0(rp_N) - \bar{h}_0 J_0(rp_0)\right]$$
$$+ \frac{1}{\delta r^2}\sum_{k=0}^{N}(\bar{h}_{k+1} - \bar{h}_k)\left[R_0(rp_{k+1}) - R_0(rp_k)\right] \qquad (4.6)$$

This is the basic formula for the Filon-trapezoidal scheme for first order Hankel transforms.

6

5. Numerical Example for the First-order Case

Let us consider the example [4]

$$\bar{h}(p) = p(1 - p^2)^{\frac{1}{2}}, \qquad 0 \leq p \leq 1 \tag{5.1}$$

$$H(r) = \frac{\pi J_1^2(\frac{r}{2})}{2r}, \qquad 0 \leq r < \infty \tag{5.2}$$

The numerical results are summarized in Table 2. As with the corresponding algorithm in Section 2, this algorithm also tends to saturate as N increases.

6. Summary

The Filon/trapezoidal scheme is not meant to be a direct competitor to the Filon/Simpson scheme. The main purpose of this quadrature scheme is to maintain a given accuracy (provided it is not too extreme) more or less uniformly, independent of the magnitude of the independent variable. An added advantage of this scheme is its speed of execution, an important aspect for such problems as beam propagation in an inhomogeneous or random medium, where the integral must be computed a large number of times. Reference is made to [5] for invention of the Hankel transforms using the sampling expansion in connection with the Filon/Simpson and the Filon/trapezoidal schemes.

References

1. R. Barakat and E. Parshall, "Numerical evaluatioin of zero-order Hankel transforms using Filon quadrature philosophy," (accepted for publication in Applied Mathematics Letters).

2. R. Barakat and B. Sandler, "Numerical evaluatioin of first-order Hankel transforms using Filon quadrature philosophy," (submitted to Applied Mathematics Letters).

3. J. Mason, "Cylindrical Bessel functions for a large range of complex arguments," Comput. Phys. Commun. 30, 1-11 (1983).

4. L. Gradshteyn and J. Ryzik, *Tables of Integrals, Series and Products* (Academic, New York, 1965). See page 688, entry 9.

5. R. Barakat, E. Parshall and B. Sandler, "Zero-order Hankel transform algorithms based on Filon quadrature philosophy for diffraction optics and beam propagation," (submitted to Journal of the Optical Society of America).

Table 1. Absolute error of Filon/trapezoidal calculations for zero-order case, example in Eqs 3.1 and 3.2.

r	$N = 100$	$N = 200$	$N = 300$	$N = 400$
6	3.727E-06	9.202E-07	4.069E-07	2.282E-07
12	3.063E-06	7.722E-07	3.447E-07	1.945E-07
18	2.099E-06	5.418E-07	2.443E-07	1.386E-07
24	1.083E-06	2.928E-07	1.343E-07	7.690E-08
30	1.901E-07	6.871E-08	3.427E-08	2.048E-08
40	8.976E-07	2.354E-07	1.068E-07	6.082E-08
50	5.229E-07	1.199E-07	5.167E-08	2.861E-08
60	2.053E-07	6.985E-08	3.411E-08	2.015E-08
80	2.544E-07	4.877E-08	1.941E-08	1.026E-08

Table 2. Absolute error of Filon/trapezoidal calculations for first-order case, example in Eqs 5.1 and 5.2.

r	$N = 100$	$N = 200$	$N = 300$	$N = 400$
6	2.764E-04	9.789E-05	5.330E-05	3.463E-05
12	2.186E-04	7.801E-05	4.260E-05	2.772E-05
18	1.769E-04	6.422E-05	3.527E-05	2.302E-05
24	1.360E-04	5.080E-05	2.822E-05	1.851E-05
30	9.406E-05	3.715E-05	2.096E-05	1.386E-05
40	1.129E-04	4.214E-05	2.331E-05	1.526E-05
50	1.029E-04	3.578E-05	1.927E-05	1.243E-05
60	7.152E-05	2.157E-06	1.093E-05	6.818E-06
70	2.863E-05	4.011E-05	9.669E-07	2.174E-07
80	1.479E-05	1.215E-05	7.884E-06	5.537E-06

NUMERICAL EVALUATION OF FOURIER INTEGRALS:

FILON QUANDRATIVE VERSUS THE FFT

1. INTRODUCTION

We are concerned with the numerical evaluation of the finlite range Fourier integral

$$f(v) = \int_a^b F(p)e^{ivp}\,dp, \qquad -\infty < v < \infty \tag{1.1}$$

assuming that $F(p)$ varies slowly compared to the complex-valued exponential term. The case where $F(p)$ is of rapid variation is analyzed by a different approach, see Section 5. For large values of $|v|$, a graph of the integrand consists of positive and negative areas of nearly equal size. The addition of these areas results in substantial loss of accuracy. Filon [1] conceived the idea of retaining Simpson's rule, but requiring that only $F(p)$ be fitted to a quadratic over the basic subinterval instead of the entire integrand $F(p)\exp(ivp)$. The fact that only $F(p)$ has to be approximated means that the number of subintervals to be taken can be relatively small in many cases of practical interest. An additional feature of the Filon quadrative philosphy is that the error incurred is relatively independent of v because the error is proportional to the derivatives of $F(p)$ itself rather than to $F(p)\exp(ivp)$.

2. FILON/SIMPSON ALGORITHM

We will now sketch in some detail the Filon/Simpson algorithm for evaluating the finite range Fourier integral (under the fundamental assumption that $F(p)$ varies slowly compared to the complex-valued exponential terms also in the integrand.

Consider the integral

$$f_k(v) = \int_a^b F(p)e^{ivp}\, dp \qquad (2.1)$$

which is effectively a double panel of three quadrature points: $p_{2k+2}, p_{2k+1}, p_{2k}$ where

$$\delta = (p_{2k+2} - p_{2k+1}) = (p_{2k+1} - p_{2k}). \qquad (2.2)$$

This panel is a subset of the larger interval $[a, b]$.

Following Filon we assume that $F(p)$ is smooth enough to be approximated by a quadratic function over the interval $p_{2k} \leq p \leq p_{2k+2}$

$$F(p) = b_1 + b_2(p - p_{2k+1}) + b_3(p - p_{2k+1})^2 \qquad (2.3)$$

where the b's are as yet unknown. Upon solving for the b's we have

$$b_1 = F_{2k}$$

$$b_2 = \frac{1}{2\delta}(F_{2k+2} - F_{2k})$$

$$b_3 = \frac{1}{2\delta^2}(F_{2k+2} - 2F_{2k+1} + F_{2k}) \qquad (2.4)$$

where $F_{2k+2} \equiv F(p_{2k+2})$, etc. The first and second derivatives of $F(p)$ are also needed. Differentiating Eq. (2.3) and then employing Eq. (2.4) yields

$$F'_{2k+2} = \frac{1}{2\delta}(3F_{2k+2} - 4F_{2k+1} + F_{2k}) \qquad (2.5)$$

$$F'_{2k} = \frac{1}{2\delta}(-F_{2k+2} + 4F_{2k+1} - 3F_{2k})$$

Finally

$$F''_{2k} = \frac{1}{\delta^2}(F_{2k+2} - 2F_{2k+1} + F_{2k}) \qquad (2.6)$$

3

Next integrate Eq. (1.1) twice by parts, the final result is

$$f_k(v) = \frac{i}{v}(F_{2k}e^{ivp_{2k}} - F_{2k+2}e^{ivp_{2k+2}})$$

$$+ \frac{i}{v^2}(F'_{2k+2}e^{ivp_{2k+2}} - F'_{2k}e^{ivp_{2k}})$$

$$+ \frac{i}{v^3}F''_{2k}(e^{ivp_{2k+2}} - e^{ivp_{2k}}) \tag{2.7}$$

Substitution of Eqs. (2.5) and (2.6) ultimately leads to

$$f_k(v) = -\frac{i}{v}(F_{2k+2}e^{ivp_{2k+2}} - F_{2k}e^{ivp_{2k}})$$

$$+ \frac{1}{2\delta v^2}(3F_{2k+2} - 4F_{2k+1} + F_{2k})e^{ivp_{2k+2}} + (F_{2k+2} - 4F_{2k+1} + 3F_{2k})e^{ivp_{2k}}$$

$$+ \frac{i}{\delta^2 v^3}(F_{2k+2} - 2F_{2k+1} + F_{2k})(e^{ivp_{2k+2}} - e^{ivp_{2k}}) \tag{2.8}$$

This expression is the basic building block of the Filon-Simpson algorithm.

To evaluate Eq. (1.1). we divide the interval of integration $[a, b]$ thusly

$$b = a + N\delta \tag{2.9}$$

where N is an even integer (as in the standard Simpson method: thus

$$f(v) = \sum_{k=0}^{(N-2)/2} f_k(v). \tag{2.10}$$

Combining terms in the summation. this eventually becomes

$$f(v) = \frac{i}{v}(F_0 e^{ivp_0} - F_N e^{ivp_N} + \sum_{k=0}^{(N-2)/2}\left(\frac{G_k}{2\delta v^2} + i\frac{H_k}{\delta^2 v^3}\right)e^{ivp_{2k}} \tag{2.11}$$

where

$$G_0 = F_2 - 4F_1 + 3F_0 \tag{2.12a}$$

$$G_k = F_{2k+2} - 4F_{2k+1} + 6F_{2k} - 4F_{2k-1} + F_{2k-2} \tag{2.12b}$$

$$G_{N-2} = 3F_N - 4F_{N-1} + F_{N-2} \tag{2.12c}$$

4

and

$$H_0 = -F_2 + 2F_1 - F_0 \tag{2.13a}$$

$$H_k = -F_{2k+2} + 2F_{2k+1} - 2F_{2k-1} + F_{2k} \tag{2.13b}$$

$$H_{N-2} = F_N - 2F_{N-1} + F_{N-2} \tag{2.13c}$$

In Eqs. (2.13b) and (2.13c), we have $0 < k < (N-2)/2$. Equation (2.11) is the basic expression for the Filon/Simpson quadrature algorithm.

A modification must be made when $v = 0$, because of the presence of v in the denominators of Eq. (2.11). Fortunately this is straightforward because

$$f(0) = \int_a^b F(p) \, dp. \tag{2.14}$$

Since $F(p)$ is assumed to be fairly smooth, the integral can be evaluated directly using a standard quadrature formula; our case, Simpson's rule.

Equation (2.11) constitutes the corrected complex exponential version of the Filon scheme found in [2]. Equation (2.11) is markedly different than the final forms of the sine and cosine versions derived by Filon [1]. He combined terms in a manner that was conducive to hand calculation techniques of his time, resulting in calculations that were independent of $F(p)$ and so could be tabulated without the knowledge of $F(p)$. However, computational abilities are considerably advanced now, so that it is unnecessary to store tables of values in order to perform the integration. Direct calculation using Eq. (2.11) by computer is not difficult. Reference is made to [3,4] for the explicit expressions of the sine and cosine versions of Filon's original analysis; they different significantly in form from Eq. (2.11) particularly with respect to the very troublesome α and β terms in the original version.

We omit any discussion of the error incurred as it is still a matter of contention [5]. However, the error is proportional to the derivatives of $F(p)$ itself rather than to the entire integrand, hence are relatively independent of the magnitude of the variable v.

3. FILON/TRAPEZOIDAL ALGORITHM

In some situations we do not require great accuracy but need to maintain a given moderate accuracy, more or less uniformly, independent of the magnitude of v. To this end we develop a trapezoidal version of the previous algorithm, the Filon/trapezoidal algorithm.

We now sketch this algorithm. Consider two points p_{k+1} and p_k which are a subset of $[a, b]$. Approximate $F(p)$ as a straight line between p_{k+1} and p_k

$$F(p) = A + Bp, \qquad p_k \leq p \leq p_{k+1} \tag{3.1}$$

Here

$$A = \frac{1}{\delta}(p_{k+1}F_k - p_k F_{k+1})$$
$$B = \frac{1}{\delta}(F_{k+1} - F_k) \tag{3.2}$$

where $\delta \equiv (p_{k+1} - p_k)$. Also

$$F' = \frac{1}{\delta}(F_{k+1} - F_k). \tag{3.3}$$

Upon integrating by parts and using the above equations, we have

$$f_k(v) = \frac{i}{v}\left(F_{k+1}e^{ivp_{k+1}} - F_k e^{ivp_k}\right)$$
$$+ \frac{1}{\delta v^2}(F_{k+1} - F_k)\left(e^{ivp_{k+1}} - e^{ivp_k}\right) \tag{3.4}$$

To evaluate the integral over $[a, b]$ i.e., Eq. (1.1), divide $[a, b]$ thusly $b = a + N\delta$ so that

$$f(v) = \sum_{k=0}^{N} f_k(v). \tag{3.5}$$

The final result is

$$f(v) = \frac{i}{v}\left(F_N e^{ivp_N} - F_0 e^{ivp_0}\right)$$
$$+ \frac{1}{\delta v^2}\sum_{k=0}^{N}(F_{k+1} - F_k)\left(e^{ivp_{k+1}} - e^{ivp_k}\right) \tag{3.6}$$

When $v = 0$, we again employ Eq. (2.14).

6

We should emphasize that the Filon/trapezoidal algorithm is not meant to be a direct competitor to the Filon/Simpson algorithm as regards great accuracy. Rather its main use is to maintain a given, but moderate accuracy, independent of the magnitude of v, when speed of execution is important.

In the special case where the limits of integration of Eq. (1.1) are symmetric (i.e., $a = -b$), then the Filon/trapezoidal algorithm can be written

$$f(v) = \left(\frac{b}{M}\right) \sum_{m=-M}^{M} H_m F\left(\frac{mb}{M}\right) e^{ivb/M}$$

where $(2M + 1)$ is now the number of quadrature points, and

$$H_m = \frac{\left(\sin \frac{vb}{M}\right)^2}{\left(\frac{vb}{M}\right)^2} \qquad \text{for} \quad m \neq \pm M$$

$$H_{+M} = \left(1 - \frac{ivb}{M} - e^{-ivb/m}\right) / \left(\frac{vb}{M}\right)^2$$

$$H_{-M} = \left(1 + \frac{ivb}{M} - e^{-ivb/m}\right) / \left(\frac{vb}{M}\right)^2$$